CRITERION OF THE VALIDITY OF STRAIN RELATIONS OF THIN SHELLS IN THE REGION OF ARBITRARY DISPLACEMENTS

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One of several approaches can be taken to determine the stress-strain state of thin shells with large elastic displacements. The author of [1] obtained relations of a shell theory with small strains and arbitrary displacements. An analysis was also made of basic methods of simplifying problems with finite displacements. Variants of the nonlinear theory of shells in a quadratic approximation were proposed in [2, 3], while in [4] equations of thin shells were derived by isolating rigid rotation in the displacements. In [5], the basic assumptions of the theory of thin shells were generalized to and refined for the case of large strains accompanied by a change in the thickness of the shell. The presence of arbitrary displacements precluded the customary use of simplifying assumptions regarding the smallness not only of the strains, but also of certain terms associated with the displacements or rotations.

The theory constructed in [1] was the starting point for many researchers in the field of the nonlinear deformation of shells. The approach taken in [1] to deriving the strain relations for arbitrary rotations and small strains is based on examining general equations of the nonlinear theory of elasticity in orthotropic curvilinear coordinates. The Kirchhoff–Love geometric hypotheses were used to obtain three nonlinear algebraic equations relative to the direction cosines ϑ, ψ, $(1 + \chi)$ of a normal to the deformed middle surface of the shell:

$$\vartheta^2 + \psi^2 + (1 + \chi)^2 = 1, \ (1 + \chi)\widehat{e}_{13} + \psi\widehat{e}_{12} + \vartheta(1 + \widehat{e}_{11}) = 0, \quad (1 + \chi)\widehat{e}_{23} + \vartheta\widehat{e}_{21} + \psi(1 + \widehat{e}_{22}) = 0, \tag{0.1}$$

where $\hat{e}_{ij}$ (i, j = 1, 2) are linear components of the strains of the middle surface; $-\widehat{e}_{i3}$ (i = 1, 2) are direction cosines of the normal to the deformed surface in a linear approximation.

With the use of the assumption that the strains could be ignored compared to unity, an approximate solution was given for system (0.1) in the form $\vartheta = -\widehat{e}_{13}(1 + \widehat{e}_{22}) + \widehat{e}_{23}\widehat{e}_{12}$, $\psi = -\widehat{e}_{23}(1 + \widehat{e}_{11}) + \widehat{e}_{13}\widehat{e}_{21}$, $\chi = \widehat{e}_{11} + \widehat{e}_{22} + \widehat{e}_{11}\widehat{e}_{22} - \widehat{e}_{12}\widehat{e}_{21}$. The authors of [6, 7] expressed doubts as to the correctness of this step in the derivation of the strain relations and instead proposed that the main criterion for evaluating the relations in [1] be the possibility of reducing them to linear expressions [8] with small displacements. In order to satisfy this criterion, in [6] the function χ was expanded into a series in powers of the displacement of the middle surface and their derivatives. Here, terms of the second order of smallness were retained in the series. As was noted in [7], such assumptions are invalid in present of arbitrary displacements. Addressing this case, the authors of [7] supplemented the strain relations with small terms of the order of $\hat{\varepsilon}_{ij}/R_j$ (R_j are the principal radii of curvature and $\hat{\varepsilon}_{ij}$ are strains of the middle surface). As was noted in [8, p. 27], "Certain authors tend to ascribe fundamental importance to the given circumstance... However, the refinement achieved here does not exceed the error of the original assumptions of the shell theory."

It can be concluded from an analysis of the above-examined studies that in regard to the derivation of variants of strain relations on the basis of the approach in [1], the question of their validity in the region of arbitrary displacements needs to be more fully investigated.

The selection of a criterion for evaluating strain relations is of both theoretical and practical importance, in connection with the possibilities of numerically solving problems of shell bending with arbitrary displacements. It should be noted that that the need for such criteria arises in evaluations of approximate relations. When adequate measures

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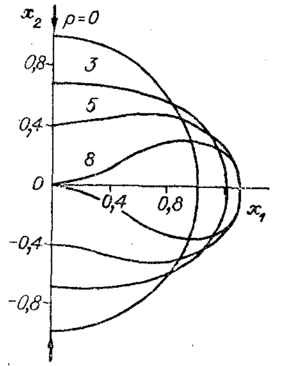

Fig. 1

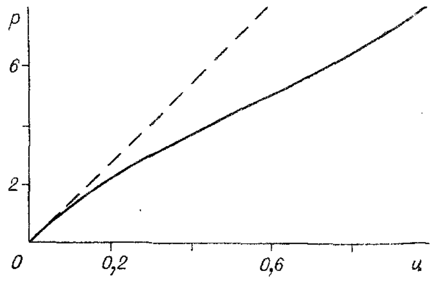

Fig. 2

of strain are employed for a certain continuum (such as in [4, 9, 10]), these relations by definition satisfy physically substantiated criteria. Since it is difficult to make practical use of the most general formulations, the approximate relations most commonly used in shell theory have certain established regions of applicability. However, even if a study is based on an initially adequate relation, certain obligatory criteria may be violated in the course of establishing various approximate expressions.

In the present study, we use the approach in [1] to derive strain relations for thin shells in vector form. These relations are then analyzed by means of criteria expressing the absence of strains and curvatures with arbitrary displacements of the shell as a rigid body. Criteria of the absence of mutual displacements of points of the shell are formulated in terms of vectors connected with the surface ("frozen vectors"). It is shown that in the principal approximation the strain relations in [1] should be supplemented by a few additional terms. A local approximation of the relations which is performed permits their use in numerical algorithms.

1. In accordance with [1], the expressions for the strains $\hat{\varepsilon}_{ij}$ and curvatures κ_{ij} of the middle surface have the form

$$\widehat{\varepsilon}_{ij} = (1/2)(\widehat{e}_{ij} + \widehat{e}_{ji} + \widehat{e}_{im}\widehat{e}_{jm}); \tag{1.1}$$

$$\varkappa_{ij} = (1/2)(k_{ij} + k_{ji} + k_{im}\widehat{e}_{jm} + k_{jm}\widehat{e}_{im}) \tag{1.2}$$

(i, j = 1, 2, and summation is carried out over m = 1, 2, 3). The components $\hat{\varepsilon}_{ij}$ and κ_{ij} form tensors of the strains and curvatures; $\hat{\varepsilon}_{12}$ and κ_{12} differ from the corresponding quantities in [1] by the multiplier 1/2. The remaining notation is the same as in [1].

To perform a geometric analysis, we write (1.1-1.2) in vector form. We introduce the vectors $\mathbf{r}_i, \mathbf{n}_i$ $(\widehat{\mathbf{r}}_i, \widehat{\mathbf{n}}_i)$ by means of the relations

$$\mathbf{r}_i = \frac{1}{A_i}\mathbf{r}_{,i}, \quad \widehat{\mathbf{r}}_i = \frac{1}{A_i}\widehat{\mathbf{r}}_{,i}, \quad \mathbf{n}_i = \frac{1}{A_i}\mathbf{n}_{,i}, \quad \widehat{\mathbf{n}}_i = \frac{1}{A_i}\widehat{\mathbf{n}}_{,i}. \tag{1.3}$$

Here, $\mathbf{r}, \mathbf{n}$ $(\widehat{\mathbf{r}}, \widehat{\mathbf{n}})$ are the position vector and normal vector of the undeformed (deformed) middle surface; A_i are the Lamé constants; $\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2$; the subscripts after the commas denote derivatives with respect to the coordinates α_i (i = 1, 2).

The vectors $\widehat{\mathbf{r}}_i$, $\widehat{\mathbf{n}}_i$ can be written directly in terms of $\hat{e}_{ij}$, k_{ij}:

$$\widehat{\mathbf{r}}_i = \mathbf{r}_i + \widehat{e}_{im}\mathbf{r}_m, \quad \widehat{\mathbf{n}}_i = \mathbf{n}_i + k_{im}\mathbf{r}_m \tag{1.4}$$

(i = 1, 2, and summation is carried out over m = 1, 2, 3; $\mathbf{r}_3 \equiv \mathbf{n}$ is the identity notation for the normal vector to the original surface).

Having calculated the following scalar products, we obtain

$$\widehat{\mathbf{r}}_i\widehat{\mathbf{r}}_j - \mathbf{r}_i\mathbf{r}_j = \widehat{e}_{ij} + \widehat{e}_{ji} + \widehat{e}_{im}\widehat{e}_{jm}; \tag{1.5}$$

$$(\widehat{\mathbf{n}}_i - \mathbf{n}_i)\widehat{\mathbf{r}}_j + (\widehat{\mathbf{n}}_j - \mathbf{n}_j)\widehat{\mathbf{r}}_i = k_{ij} + k_{ji} + k_{im}\widehat{e}_{jm} + k_{jm}\widehat{e}_{im}, \tag{1.6}$$

where we have used the relation $\mathbf{r}_i\mathbf{r}_j = \delta_{ij}$ (δ_{ij} is the Kronecker symbol).

Comparing (1.5-1.6) with (1.1-1.2), we find the vector form of $\hat{\varepsilon}_{ij}$, κ_{ij}:

$$\widehat{\varepsilon}_{ij} = (1/2)(\widehat{\mathbf{r}}_i\widehat{\mathbf{r}}_j - \mathbf{r}_i\mathbf{r}_j); \qquad (1.7)$$

$$\varkappa_{ij} = (1/2)(\widehat{\mathbf{n}}_i\widehat{\mathbf{r}}_j + \widehat{\mathbf{n}}_j\widehat{\mathbf{r}}_i - \mathbf{n}_i\widehat{\mathbf{r}}_j - \mathbf{n}_j\widehat{\mathbf{r}}_i). \qquad (1.8)$$

In accordance with [1], for the strains ε_{ij} in a layer located the distance z from the middle surface we have $\varepsilon_{ij} = \varepsilon_{ij} + z\varkappa_{ij} + z^2\nu_{ij}$ or, in vector form,

$$\varepsilon_{ij} = (1/2)[(\widehat{\mathbf{R}}_i - \mathbf{R}_i)\mathbf{r}_j + (\widehat{\mathbf{R}}_j - \mathbf{R}_j)\mathbf{r}_i + (\widehat{\mathbf{R}}_i - \mathbf{R}_i)(\widehat{\mathbf{R}}_j - \mathbf{R}_j)]; \qquad (1.9)$$

$$\widehat{\mathbf{R}}_i = \widehat{\mathbf{r}}_i + z\widehat{\mathbf{n}}_i, \ \mathbf{R}_i = \mathbf{r}_i + z\mathbf{n}_i. \qquad (1.10)$$

Since the abbreviated relations $\varepsilon_{ij} = \widehat{\varepsilon}_{ij} + z\varkappa_{ij}$, are usually used for thin shells, we will henceforth ignore the values of ν_{ij} [1].*

2. We will assume that **a** and **b** are vectors continuously connected with the surface ("frozen in the surface"). With the displacement of the surface of the shell as a rigid body without strains and curvatures, the vectors **a** and **b** also undergo a certain displacement in space and assume the new positions $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{b}}$. However, there is no accompanying change in the position of the vectors relative to each other in this case; thus, there remains the scalar product

$$\widehat{\mathbf{a}}\widehat{\mathbf{b}} = \mathbf{ab}. \qquad (2.1)$$

In the special case **b** = **a**, the property of conservation of the length of the vector $\widehat{\mathbf{a}}^2 = \mathbf{a}^2$ follows from (2.1).

Among the vectors that are continuously connected with the the surface are derivatives (of any order) of the position-vector with respect to the curvilinear coordinates and their linear combinations: $\mathbf{r}_i$, $\mathbf{n}_i$, $\mathbf{R}_i$, etc. The "frozen" state of these vectors results in conservation of the coefficients of the first and second quadratic forms of the surface with rigid-body displacements, since these coefficients are expressed in terms of the first and second derivatives of the position-vector. It should be noted that the position-vector itself is not among the frozen vectors. Thus, in accordance with (2.1), with rigid-body displacements we have

$$\widehat{\mathbf{r}}_i\widehat{\mathbf{r}}_j = \mathbf{r}_i\mathbf{r}_j, \ \widehat{\mathbf{n}}_i\widehat{\mathbf{r}}_j = \mathbf{n}_i\mathbf{r}_j, \ \widehat{\mathbf{R}}_i\widehat{\mathbf{R}}_j = \mathbf{R}_i\mathbf{R}_j. \qquad (2.2)$$

Here, (1.7-1.9) show that $\widehat{\varepsilon}_{ij} = 0$, $\varkappa_{ij} = (1/2)[\mathbf{n}_i(\mathbf{r}_j - \widehat{\mathbf{r}}_j) + \mathbf{n}_j(\mathbf{r}_i - \widehat{\mathbf{r}}_i)]$, $\varepsilon_{ij} = (1/2)[(\widehat{\mathbf{R}}_i - \mathbf{R}_i)(\mathbf{r}_j - \mathbf{R}_j) + (\widehat{\mathbf{R}}_j - \mathbf{R}_j)(\mathbf{r}_i - \mathbf{R}_i)]$.

It should be noted that that Eq. (1.9) is the determining equation in the given case, since the formulas for κ_{ij} are derived from (1.9) by means of (1.10) and are found as coefficients with equal powers of z [1]. In the calculation of the strains at the distance z from the middle surface, the position-vector and its derivatives in the original position should also be considered independent of z. As a result, by replacing $\mathbf{r}_i$ by $\mathbf{R}_i(z)$ in (1.9) in accordance with (1.10) and by reducing similar terms, we arrive at the expression

$$\varepsilon_{ij} = (1/2)(\widehat{\mathbf{R}}_i\widehat{\mathbf{R}}_j - \mathbf{R}_i\mathbf{R}_j), \qquad (2.3)$$

which is consistent with the definition of the components of the Green strain tensor referred to a metric surface.

In accordance with (2.2), in the presence of rigid-body displacements we have $\varepsilon_{ij} = 0$. Inserting (1.10) into (2.3) and grouping terms with the first powers of z, we find

$$\varkappa_{ij} = (1/2)(\widehat{\mathbf{n}}_i\widehat{\mathbf{r}}_j + \widehat{\mathbf{n}}_j\mathbf{r}_i - \mathbf{n}_i\mathbf{r}_j - \mathbf{n}_j\mathbf{r}_i). \qquad (2.4)$$

Using Eqs. (2.2), we obtain $\kappa_{ij} = 0$. Substitution of (1.4) into (2.4) leads to the following relations (with summation over m = 1, 2, 3)

$$\varkappa_{ij} = \frac{1}{2}\left(k_{ij} + k_{ji} + k_{im}\widehat{e}_{jm} + k_{jm}\widehat{e}_{im} + \frac{1}{R_j}\widehat{e}_{ij} + \frac{1}{R_i}\widehat{e}_{ji}\right). \qquad (2.5)$$

*It should be noted that the contribution of the terms with $z^2\nu_{ij}$ may be significant in the case of severe local bending in the plastic region.

Here, R_i are the principal radii of curvature obtained with the expansion of the expression $\mathbf{n}_i\mathbf{r}_j = (1/R_i)\delta_{ij}$. Let us evaluate the additional terms in (2.5). Since [in accordance with (1.4)] the sum $\delta_{ij} + \hat{e}_{ij}$ coincides to within terms of the order of the strains with the direction cosines of the vector $\widehat{\mathbf{r}}_i$, then for any displacements we have

$$-1 \leqslant \delta_{ij} + \widehat{e}_{ij} \leqslant 1. \tag{2.6}$$

With i = j, we find from (2.6) that $-2 \leqslant \widehat{e}_{ii} \leqslant 0$, while at i ≠ j we have $-1 \leqslant \widehat{e}_{ij} \leqslant 1$. The boundary values of these inequalities may change by an amount which is of the order of magnitude of the strains. Expressions (2.5) differ from the strain relations in [6, 7].

Let us examine (2.5) in the case of cylindrical bending of an inextensible ($\hat{\varepsilon}_{ij} = 0$) curved strip of radius $R_1(\alpha_1)$. The quantity $\varkappa_{11}$ will be nontrivial. As the parameter α_1, we take the length of an arc of the middle surface in the undeformed state, when $A_1 = 1$. Of $\hat{e}_{ij}$, k_{ij}, nontrivial quantities will be $\widehat{e}_{11} = \widehat{u}_{,1} + \widehat{w}/R_1$, $\widehat{e}_{13} = \widehat{w}_{,1} - \widehat{u}/R_1$, $k_{11} = -\widehat{e}_{13,1} + \widehat{e}_{11}/R_1$, $k_{13} = e_{11,1} + \widehat{e}_{13}/R_1$. Here, $\hat{u}$ and $\hat{w}$ are the longitudinal and transverse displacements. The subscript after the comma denotes a derivative with respect to α_1. With allowance for $\hat{\varepsilon}_{11} = 0$, expression (2.5) for $\varkappa_{11}$ takes the form $\varkappa_{11} = -(1 + \widehat{e}_{11})\widehat{e}_{13,1} + \widehat{e}_{13}\widehat{e}_{11,1}$, which agrees with the exact formula for the change in the curvature of a plane curve.

3. Let us examine the possibility of using the vector analog of Eq. (1.1) and (1.5):

$$\widehat{\varepsilon}_{ij} = (1/2)(\widehat{\mathbf{r}}_i\widehat{\mathbf{r}}_j - \mathbf{r}_i\mathbf{r}_j), \quad \varkappa_{ij} = (1/2)(\widehat{\mathbf{n}}_i\widehat{\mathbf{r}}_j + \widehat{\mathbf{n}}_j\widehat{\mathbf{r}}_i - \mathbf{n}_i\mathbf{r}_j - \mathbf{n}_j\mathbf{r}_i). \tag{3.1}$$

In the neighborhood of the regular point O of an arbitrary surface, we have the following approximation relations [11] (with the origin of the coordinates α_1, α_2 placed at point O)

$$\mathbf{r} = \mathbf{r}^p + \zeta\mathbf{n}^p, \quad A_i^2 \simeq (\mathbf{r}^p_{,i})^2 = 1, \quad \zeta \simeq \frac{1}{2}\left(\frac{1}{R_1}\alpha_1^2 + \frac{1}{R_2}\alpha_2^2\right)$$

($\mathbf{r}^p(\alpha_1, \alpha_2)$ is the position vector on the point tangent to the surface at the given point; $\mathbf{n}^p$ is a normal to the plane; $\zeta(\alpha_1, \alpha_2)$ is the function giving the form of the surface). Using a similar approximation for the deformed surface [12] and allowing for (1.3), we represent (3.1) in the form

$$\widehat{\varepsilon}_{ij} \simeq (1/2)(\widehat{\mathbf{r}}^p_{,i}\widehat{\mathbf{r}}^p_{,j} - \mathbf{r}^p_{,i}\mathbf{r}^p_{,j}); \tag{3.2}$$

$$\varkappa_{ij} \simeq (1/2)(\widehat{\mathbf{n}}_{,i}\widehat{\mathbf{r}}^p_{,j} + \widehat{\mathbf{n}}_{,j}\widehat{\mathbf{r}}^p_{,i} - \mathbf{n}_{,i}\mathbf{r}^p_{,j} - \mathbf{n}_{,j}\mathbf{r}^p_{,i}). \tag{3.3}$$

The terms omitted from (3.2-3.3) are of the order $O(\alpha^2)$ [12], where $\alpha = (\alpha_1^2 + \alpha_2^2)^{1/2}$ is the radius of the neighborhood of the point of tangency. Equation (3.3) can be written in the following equivalent form with allowance for the equality $\mathbf{r}^p_{,ij} = 0$:

$$\varkappa_{ij} = (1/2)(\Theta_{i,j} + \Theta_{j,i}), \quad \Theta = \widehat{\mathbf{n}}\,\widehat{\mathbf{r}}^p_{,i} - \mathbf{n}\mathbf{r}^p_{,i}, \tag{3.4}$$

which makes it similar in meaning to the analogous expressions in the linear theory of plates and shells. However, here we calculate Θ_i from a nonlinear formula. It can also be shown [12] that relations (3.2), (3.4) are invariant under a transformation of the coordinates α_1, α_2. Within the present context, the tangent plane to the initial and deformed surfaces $\mathbf{r}^p_{,i}$ is a "frozen vector." This means that the rigid-body displacements

$$\widehat{\mathbf{r}}^p_{,i}\widehat{\mathbf{r}}^p_{,j} = \mathbf{r}^p_{,i}\mathbf{r}^p_{,j}, \quad \widehat{\varepsilon}_{ij} = 0, \quad \widehat{\mathbf{n}}\,\widehat{\mathbf{r}}^p_{,i} = \mathbf{n}\mathbf{r}^p_{,i}, \quad \varkappa_{ij} = 0.$$

Thus, (3.2-3.4) can be used as measures of strains and curvatures with arbitrary displacements.

The approximate deformation relations are convenient to use in direct numerical methods, such as the finite-element method. Here, information on the radius-vector and normal vector at a finite number of points is used as the initial geometric characteristics.

Figure 1 shows the solution of the problem of the symmetric deformation of a half-ring referred to the coordinates x_1, x_2. The ring is of the radius R = 1 and is loaded by two concentrated forces with the value p^+. Figure 2 shows the dependence of the load $p = 2p^+R^2/D$ on the displacement $u = u^+/(2R)$ (u^+ is the mutual displacement of the points of application of the forces and D is bending stiffness). The dashed line shows the linearized dependence. The solution was obtained by the finite-element method on the basis of (3.2), (3.4) for $\hat{\varepsilon}_{11}$, κ_{11}. The nonlinear deformation characteristic and the equilibrium modes agree well with the exact solution found by the method of elliptic integrals [13].

The authors of [12, 16, 17] determined the strain state of shells by a method based on relations (3.2), (3.4), canonical energy forms [14], and variational formulas of vector algebra [15].

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